Absolutes of Hausdorff spaces and cardinal invariants F_θ and t_θ

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Abstract

This article extends the recent study of the cardinal functions F_θ and t_θ for H-closed Urysohn spaces and the research of I. Bandlov and V.I. Ponomarev on tightness type of absolutes. In particular, basic results are obtained and used to study the relationships among the cardinal functions t, t_θ, F and F_θ in the context of Iliadis and Banaschewski absolutes of Hausdorff spaces.

Keywords: *closure*, *θ-closure*, *free sequence*, *θ-free sequence*, $t(X)$, $t_\theta(X)$, $F(X)$, $F_\theta(X)$, *compact spaces*, *H-closed spaces*, *Urysohn spaces*, *absolutes of Hausdorff spaces*.

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1 Introduction

Absolutes can be traced to the fundamental papers by M.H. Stone ([23, 24]). Absolutes received a major boost in research by A. Gleason ([15]) in 1958 quickly followed by the major studies of B. Banaschewski, J. Flachsmeyer, S. Iliadis, J. Mioduszewski, V.I. Ponomarev, L. Rudolf and L.B. Shapiro ([5, 13, 17, 19, 20, 21]). The tightness type of cardinal invariants were investigated for absolutes by I. Bandlov and V.I. Ponomarev ([6]) in 1980.

This paper continues this line of research for absolutes using two relatively new cardinal functions F_θ and t_θ. In particular, we obtain basic results connecting t, t_θ, F, and F_θ and evaluate these four invariants for the Iliadis and Banaschewski absolutes for the well-known spaces of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{J}$, $\mathbb{R}$, $\mathbb{I}^\kappa$, *and* $\mathbb{D}^\kappa$.

2 Notations, terminologies and basic properties

Throughout this paper X will denote a Hausdorff space and $\tau(X)$ the topology on X. Our notation and terminology are mainly as in [12] (for general topological notions), [2], [16], [18] (for cardinal functions) and [22] (for H-closed spaces, H-closed extensions and absolutes of Hausdorff spaces).

Here are a few basic definitions:

- For a space X, recall that $\tau(X)(s)$ is the topology generated by the base $RO(X) = \{U \in \tau(X) : U = int_X(cl_X(U))\}$ (semiregularization of X). A space X is *semiregular* if its topology $\tau(X)$ coincides with the topology $\tau(X)(s)$ and we denote it by $X(s)$ (or X_s).
Clearly, *every T_3-space X is semiregular* (the converse is not true).

- A function $f : X \to Y$ is *θ-continuous* if for each $x \in X$ and open neighborhood V of $f(x)$, there is an open neighborhood U of x such that $f(cl_X(U)) \subseteq cl_Y(V)$. It easy to see that *every continuous function is θ-continuous* (the converse is not true).

- A surjection $f : X \to Y$ is *irreducible* if for each closed set $A \subseteq X$, if $A \neq X$, then $f(A) \neq Y$. Equivalently, f is *irreducible* iff for each nonempty open set $U \in \tau(X)$, there is $y \in Y$ such that $f^{\leftarrow}(y) \subseteq U$.

- A space X is *H-closed* if X is closed in every Hausdorff space containing X as a subspace. Equivalently, X is *H-closed* if every open cover $\mathcal{U}$ of X has a finite subfamily $\mathcal{V}$ whose union is dense in X (i.e. $X \subseteq cl_X(\bigcup_{V\in\mathcal{V}} V)$).
We need this well-known result (see in [22]):

X is H-closed Urysohn iff X_s is compact Hausdorff.

- A space X is *extremally disconnected* (or *ED* for short) if the closure of every open set is open or, equivalently, if the closure of every open subset is clopen in X, i.e., in symbol $CLOP(X) = RO(X)$.
It is easy to verify that

X is ED iff X_s is ED and semiregular.

- ([12]) Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two collections of spaces, $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$ their product spaces and let $\{f_i\}_{i\in I}$ be a family of functions $f_i \in F(X_i, Y_i)$. The *product function* $f = \prod_{i\in I} f_i$ is defined by $\Pi_i^Y \circ f = f_i \circ \Pi_i^X$ for each $i \in I$ (where $\Pi_i^X : X \to X_i$ and $\Pi_i^Y : Y \to Y_i$ are respectively the i-th projection functions of X onto X_i and Y onto Y_i).

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\Pi_i^X \downarrow & & \downarrow \Pi_i^Y \\
X_i & \xrightarrow{f_i} & Y_i
\end{array}$$

Then, the product function is explicitely defined by:

$$f\,(< x_i >_{i\in I}) = \left(\prod_{i\in I} f_i\right)(< x_i >_{i\in I}) = < f_i(x_i) >_{i\in I}$$

for each $x =< x_i >_{i\in I}\in X$.
It is well known that if for each $i \in I$, f_i is continuous (or θ-continuous), then $f = \prod_{i\in I} f_i$ is continuous (or θ-continuous) and conversely.

- ([22]) For a space X, let $X^* = X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$. Let κX be the set X^* with the topology generated by the base $\tau(X) \cup \{U \cup \{\mathcal{U}\} : U \in \mathcal{U} \in X^* \setminus X\}$, and σX be the set X^* with the topology generated by the base $\{o(U) : U \in \tau(X)\}$ where $o(U) = U \cup \{\mathcal{U} \in X^* \setminus X : U \in \mathcal{U}\}$. Both spaces κX and σX are H-closed extensions of X. κX is called the *Katětov H-closed extension of* X and σX is said the *Fomin H-closed extension of* X. The identity function $id : \kappa X \to \sigma X$ is continuous. The remainder of κX ($= \kappa X \setminus X$) is *discrete* and *closed* in κX, and the remainder of σX ($= \sigma X \setminus X$) is a *zero-dimensional subspace* of σX. If X is a Tychonoff space, then $\kappa X \geq_X \sigma X \geq_X \beta X$ where βX denote the Stone-Čech compactification of X. When X is Tychonoff, $\kappa X = \beta X$ iff X is compact and $\sigma X = \beta X$ iff every closed nowhere dense subset of X is compact. Also, we have that $(\kappa X)_s = (\sigma X)_s = \beta X$.

- ([22]) Let X be a space and θX (called *the Stone space generated by* $RO(X)$ or *the Gleason cover of* X) denote the set of all open ultrafilters on X. For $U \in \tau(X)$ let $oU = \{\mathcal{U} \in \theta X : U \in \mathcal{U}\}$ and the topology on θX generated by $\{oU : U \in \tau(X)\}$ is ED and compact Hausdorff. The subspace $EX = \{\mathcal{U} \in \theta X : a(\mathcal{U}) \neq \emptyset\}$ (called *the Iliadis absolute of* X) is dense, ED and T_3 (hence 0-dimensional). We define $k_X : EX \to X$ by $k_X(\mathcal{U}) = p$ where $a(\mathcal{U}) = \{p\}$. The function k_X is *onto*, *perfect*, *irreducible* and *θ-continuous*. Also, the function k_X is continuous if and only if X is T_3. Note that $EX = \bigcup_{p\in X} k_X^{\leftarrow}(p)$.
In general, $\{oU \cap k_X^{\leftarrow}(V) : U, V \in \tau(X)\}$ is a base for a topology on EX (finer than $\tau(EX)$). The set EX with this finer topology is denoted by PX (called *the Banaschewski absolute of* X). The map $\Pi_X : PX \to X$ defined by $\Pi_X(\mathcal{U}) = k_X(\mathcal{U})$ is *onto*, *perfect*, *irreducible* and *continuous*. The space PX is ED but may not be T_3 (hence not 0-dimensional). Also, $\tau(PX)(s) = \tau(EX)$ and when X is T_3, $PX = EX$.

This following fact is well-known:

X is H-closed iff EX is compact iff PX is H-closed Urysohn.

For the *Katětov H-closed extension $\kappa\omega$ of ω*, note that $P(\kappa\omega) = \kappa\omega$ and $E(\kappa\omega) = (P(\kappa\omega))_s = (\kappa\omega)_s = \beta\omega$.
For the *Fomin H-closed extension $\sigma\omega$ of ω*, note that $P(\sigma\omega) = \sigma\omega = P(\beta\omega) = \beta\omega$ and $E(\sigma\omega) = (P(\sigma\omega))_s = (\beta\omega)_s = \beta\omega$.

Definition 1. For $x \in X$,
$t(x, X) = \min\{\kappa : \forall\ A \subset X$ with $x \in \overline{A}\ \exists\ B \subset A$ s.t. $|B| \leq \kappa$ and $x \in \overline{B}\}$ is called the *tightness of X at x*.
$t(X) = \sup_{x\in X}\{t(x, X)\} + \omega$ is called the *tightness of X*.

$t_\theta(x, X) = \min\{\kappa : \forall\ A \subset X$ with $x \in cl_\theta(A)\ \exists\ B \subset A$ s.t. $|B| \leq \kappa$ and $x \in cl_\theta(B)\}$ is called the *θ-tightness of X at x*.
$t_\theta(X) = \sup_{x\in X}\{t_\theta(x, X)\} + \omega$ is called the *θ-tightness of X*.

Definition 2.
A sequence $(x_\alpha : \alpha \in \mu)$ in a space X is called a *free sequence of length μ* if for every $\alpha \in \mu$ we have

$$cl_X\{x_\beta : \beta < \alpha\} \cap cl_X\{x_\beta : \beta \geq \alpha\} = \emptyset.$$

A sequence $(x_\alpha : \alpha \in \mu)$ in a space X is called a *θ-free sequence of length μ* if for every $\alpha \in \mu$ we have

$$cl_\theta\{x_\beta : \beta < \alpha\} \cap cl_\theta\{x_\beta : \beta \geq \alpha\} = \emptyset.$$

We define:
$F(X) = \sup\{\mu :$ there is a free sequence of length μ in $X\} + \omega$.
$F_\theta(X) = \sup\{\mu :$ there is a θ-free sequence of length μ in $X\} + \omega$.

Here are some basic results that we will use throughout the paper:

Proposition 3. Let X be a space and $Y \subseteq X$ as subspace.

(a) $t_\theta(X) = t_\theta(X_s)$ and $F_\theta(X) \leq F(X)$;

(b) If σ is a topology in X such that $\sigma \supseteq \tau(X)$, then $F(X) \leq F(X, \sigma)$ (in particular $F(X_s) \leq F(X)$ and $F(EX) \leq F(PX)$);

(c) $t(Y) \leq t(X)$;

(d) If Y is closed in X, then $F(Y) \leq F(X)$;

(e) If X is T_3, then $t_\theta(X) = t(X)$ and $F_\theta(X) = F(X)$;

(f) If X is Hausdorff, then $F(X) \leq L(X)t(X)$.

Recall the following result:

Theorem 4.
(a) [1] *If X is compact Hausdorff, then $F(X) = t(X)$;*
(b) [7] *If X is H-closed Urysohn, then*

$$F_\theta(X) = F_\theta(X_s) = F(X_s) = t_\theta(X) = t_\theta(X_s) = t(X_s).$$

Note 5.
In [7], we constructed an H-closed space H for which $F_\theta(H) < t_\theta(H)$.

Also, recall the following result proved by B. Balcar and F. Franěk:

Theorem 6. ([4]) *If X is compact ED, then $|X| = w(X) = t(X)$ and there is a continuous surjection $f : X \to \mathbb{D}^{w(X)}$.*

Now, we start with these straightforward results:

Lemma 7. ([22]) Let X and Y be spaces, $A \subseteq X$ and $f : X \to Y$ θ-continuous. Then, $f(cl_\theta(A)) \subseteq cl_\theta(f(A))$.

Lemma 8. Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two collections of spaces, $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$. Suppose $f = \prod_{i\in I} f_i : X \to Y$ where $f_i : X_i \to Y_i$ (for each $i \in I$) are surjections. Then, f is irreducible if and only if f_i is irreducible for each $i \in I$.

Proof. Let $f : X \to Y$ be irreducible and, fixed $i \in I$, we want to show that $f_i : X_i \to Y_i$ is irreducible. So, let $i \in I$ and a nonempty open set U_i in X_i. Then, $U = U_i \times \prod_{j\neq i} X_j$ is a nonempty open set in X. By hypothesis, there exists $y =< y_i >_{i\in I} \in Y$ such that $f^\leftarrow(y) \subseteq U$. So, for each $i \in I$, $f_i^\leftarrow(y_i) \subseteq U_i$. Thus, $f_i : X_i \to Y_i$ is irreducible for each $i \in I$.
Conversely, suppose $f_i : X_i \to Y_i$ is irreducible for each $i \in I$ and we want to show that $f : X \to Y$ is irreducible. So, let U be a nonempty open set in X. We want to find $y \in Y$ such that $f^\leftarrow(y) \subseteq U$. Now, there are a finite set $J \subseteq I$ and a nonempty open set U_i in X_i for each $i \in I$ such that $\emptyset \neq \prod_{i\in J} U_i \times \prod_{i\in I\setminus J} X_i \subseteq U$. By hypothesis, for each $i \in I$, there is $y_i \in Y_i$ such that $f_i^\leftarrow(y_i) \subseteq U_i$ and for $i \notin J$ select a point $y_i \in Y_i$. For $y =< y_i >_{i\in I}$, $f^\leftarrow(y) = \prod_{i\in I} f_i^\leftarrow(y_i) \subseteq \prod_{i\in J} U_i \times \prod_{i\in I\setminus J} X_i \subseteq U$. Thus, $f : X \to Y$ is irreducible. $\square$

Lemma 9. Let X and Y be spaces, $p \in X$ and $f : X \to Y$ be a perfect, irreducible, θ-continuous surjection. Then, p is isolated in X if and only if $f(p)$ is isolated in Y.

Proof. Suppose p is isolated in Y and, since f is irreducible, it follows that $f^{\leftarrow}(f(p)) = \{p\}$. As, $X \setminus \{p\}$ is closed, $f(X \setminus \{p\}) = f(X \setminus f^{\leftarrow}(f(p))) = f(X) \setminus \{f(p)\} = Y \setminus \{f(p)\}$ is closed. So, $\{f(p)\}$ is open and $f(p)$ is isolated in Y. Conversely, suppose $f(p)$ is isolated in Y and, since f is θ-continuous, there is an open set $U \in \tau(X)$ such that $p \in U$ and $f(cl_X U) \subseteq cl_Y\{f(p)\} = \{f(p)\}$. So, $U \setminus \{p\}$ is open and if $U \setminus \{p\} \neq \emptyset$, there is $y \in Y$ such that $f^{\leftarrow}(y) \subseteq U \setminus \{p\}$. As $f(U) = \{f(p)\}$, $y = f(p)$ implying $p \in f^{\leftarrow}(f(p)) \subseteq U \setminus \{p\}$: a contradiction. Thus, $U = \{p\}$ is open and p is isolated in X. □

Note 10.
(*a*) *Example 15* in [7] shows that there is an H-closed and Urysohn space X such that $F_\theta(X) < F(X)$.
(*b*) In general, neither $t(X) \leq t_\theta(X)$ nor $t_\theta(X) \leq t(X)$. In fact, in [7], the space in *Example 11* shows that $t_\theta(X) < t(X)$ and the space in *Example 12* shows that $t(X) < t_\theta(X)$.

3 Main results and examples

First, we examine some of the basic properties concerning cardinal functions F_θ and t_θ on absolutes:

Theorem 11. *For a Hausdorff space X we have that*

(a) $F(PX) \geq F_\theta(PX) = F(EX) = F_\theta(EX) \geq F_\theta(X)$;

(b) $t(PX) = t_\theta(PX) = t(EX) = t_\theta(EX) \geq t_\theta(X)$.

Proof.
(*a*) Let $(x_\alpha)_{\alpha \in \mu}$ be a θ-free sequence in X and choose $y_\alpha \in k_X^{\leftarrow}(x_\alpha)$ and $\beta < \mu$.
Then, by *Lemma 7*, we have that

$$k_X\left(cl_\theta\{y_\alpha\}_{\alpha \leq \beta}\right) \subseteq cl_\theta\{k_X(y_\alpha)\}_{\alpha \leq \beta} = cl_\theta\{x_\alpha\}_{\alpha \leq \beta}$$

and also

$$k_X\left(cl_\theta\{y_\alpha\}_{\alpha > \beta}\right) \subseteq cl_\theta\{k_X(y_\alpha)\}_{\alpha > \beta} = cl_\theta\{x_\alpha\}_{\alpha > \beta}$$

Moreover, $cl_\theta\{x_\alpha\}_{\alpha \leq \beta} \cap cl_\theta\{x_\alpha\}_{\alpha > \beta} = \emptyset$ and so $cl_\theta\{y_\alpha\}_{\alpha \leq \beta} \cap cl_\theta\{y_\alpha\}_{\alpha > \beta} = \emptyset$.
Thus $F_\theta(X) \leq F_\theta(EX) = F(EX) = F_\theta(PX) \leq F(PX)$.
(*b*) Let $A \subseteq X$ and $p \in cl_\theta(A)$; we have that $k_X^{\leftarrow}(p)$ is compact and assume $k_X^{\leftarrow}(p) \cap cl_{EX} k_X^{\leftarrow}(A) = \emptyset$ and this means that there exists a clopen set U such that $k_X^{\leftarrow}(p) \subseteq U$ and $U \cap k_X^{\leftarrow}(A) = \emptyset$; then $k_X(U) \cap A = \emptyset$ and so $k_X(U) = cl_X(V)$ with $p \in V$. Thus $p \notin cl_\theta(A)$ and this means that there exists $q \in k_X^{\leftarrow}(p) \cap cl_{EX} k_X^{\leftarrow}(A)$ and then $q \in cl_{EX} k_X^{\leftarrow}(A)$. So, there is $B \subset k_X^{\leftarrow}(A)$ such that $|B| \leq t(EX) = t_\theta(EX)$ and $q \in cl_{EX}(B) = cl_\theta(B)$.

Now we have $k_X(q) \in cl_\theta(k_X(B))$ where $p = k_X(q)$ and $k_X(B) \subseteq A$; so, $|k_X(B)| \leq |B| \leq t(EX)$.
Thus $t_\theta(X) \leq t_\theta(EX) = t(EX) = t_\theta(PX) = t(PX)$. $\square$

Note 12.
(a) ([7] - *Ex. 12*) $\omega = t(\kappa\omega) < \mathfrak{c} = t_\theta(\kappa\omega) = F_\theta(\kappa\omega) < 2^{\mathfrak{c}} = F(\kappa\omega)$.
(b) ([16] - *Ex. 7.22*) As $\sigma\omega = \beta\omega$, $\mathfrak{c} = t(\sigma\omega) = F(\sigma\omega) = F_\theta(\sigma\omega) = t_\theta(\sigma\omega)$.

Proposition 13. Let X be a Hausdorff space and Y be ED.

(a) $|X| \leq |EX| \leq 2^{2^{|X|}}$ and $|X| \leq |PX| \leq 2^{2^{|X|}}$;

(b) EX is ED semiregular and EY is homeomorphic to Y_s;

(c) PX is ED and PY is homeomorphic to Y;

(d) If D is a dense set of isolated points in X, then $D \subseteq EX \subseteq \beta D$;

(e) There is a discrete space D such that $|D| \leq d(Y_s)$ and Y_s can be embedded in βD;

(f) A countable subset A of Y_s is C^*-embedded in Y_s. In particular, if B is an infinite compact subspace of Y_s, then B contains a copy of $\beta\omega$ (i.e. contains a subset $C \simeq \beta\omega$);

(g) If $\beta\omega \hookrightarrow Y$, then $t(Y) \geq \mathfrak{c}$ and $F(Y) \geq \mathfrak{c}$;

(h) If $p \in X$, then $\tau(EX)|_{k_X^{\leftarrow}(p)} = \tau(PX)|_{\Pi_X^{\leftarrow}(p)}$.

Proof.
(a) As $k_X : EX \to X$ is onto, $|X| \leq |EX| = |PX|$.
Also, as $\{\mathcal{U} : \mathcal{U}$ is a fixed open ultrafilter on $X\} \subseteq \mathcal{P}(\mathcal{P}(X))$, then $|PX| = |EX| \leq |\mathcal{P}(\mathcal{P}(X))| = 2^{2^{|X|}}$.
For the facts *(b)*, *(c)*, *(d)*, *(e)* and *(f)* we refer the reader to [22].
(g) We have that $\mathfrak{c} = t(\beta\omega) \leq t(Y)$ and $\mathfrak{c} = F(\beta\omega) \leq F(Y)$.
(h) Both $k_X^{\leftarrow}(p)$ and $\Pi_X^{\leftarrow}(p)$ are compact subspaces in the same set. As $\tau(EX) \subseteq \tau(PX)$, $\tau(k_X^{\leftarrow}(p)) \subseteq \tau(\Pi_X^{\leftarrow}(p))$. Since compact Hausdorff spaces are minimal Hausdorff, it follows that $\tau(k_X^{\leftarrow}(p)) = \tau(\Pi_X^{\leftarrow}(p))$. $\square$

By *Proposition 13(a)*, it is natural to ask if the following inequalities are true or not:

* If X is Hausdorff, then $F_\theta(X) \leq F_\theta(EX) \leq 2^{2^{F_\theta(X)}}$.
* If X is Hausdorff, then $t_\theta(X) \leq t_\theta(EX) \leq 2^{2^{t_\theta(X)}}$.

Proposition 14. Let E be an ED, semiregular space and D a discrete subspace of E such that $|D| = d(E)$. Then,

$$t(E) \leq t(\beta E) \leq t(\beta D) \leq w(\beta D) \leq 2^{|D|} = 2^{d(E)}.$$

Proof. By *Proposition 13(e)*, $\beta E \hookrightarrow \beta D \setminus D$ and $t(\beta E) \leq t(\beta D)$. As, $E \subseteq \beta E$, then $t(E) \leq t(\beta E)$. It always true that $t(\beta D) \leq w(\beta D)$. Finally, by *3.3(b)* in [16], $w(\beta D) \leq 2^{d(\beta D)} = 2^{|D|} = 2^{d(E)}$. $\square$

Proposition 15. Let X be a H-closed space with a dense set D of isolated points. Then,

(a) $ED = D$ is dense in EX;

(b) $EX \equiv_{ED} \beta D$;

(c) PX is the set βD with a finer topology σ and $\tau(\beta D) \subseteq \sigma \subseteq \tau(\kappa D)$.

Proof. For the facts (a) and (b) we refer the reader to [22].
(c) PX is EX (and by (b), $\equiv_{ED}$ βD) with a finer topology, i.e, $\tau(PX) \supseteq \tau(EX)$, and $\tau(PX)(s) = \tau(EX)$. That is, we can consider PX as βD with a topology σ such that $\sigma \supseteq \tau(\beta D)$ and $\sigma(s) = \tau(\beta D)$. Also, κD is βD with a finer topology such that $\tau(\kappa D) \supseteq \tau(\beta D)$ and $\tau(\kappa D)(s) = \tau(\beta D)$. By *7.7* in [22], there is a continuous bijection from κD to $(\beta D, \sigma)(= PX)$ that leaves the points of D fixed. Thus, $\sigma \subseteq \tau(\kappa D)$. $\square$

Example 16. The inequalities $F_\theta(EX) \leq 2^{2^{F_\theta(X)}}$ and $t_\theta(EX) \leq 2^{2^{t_\theta(X)}}$ are false. To show this, let D be infinite discrete space of cardinality κ and $X = \alpha D$ be one-point compactification of D. Note that $t_\theta(X) = t(X) = F_\theta(X) = F(X) = \omega$. Also, $EX = E(\alpha D) = \beta D$ and $t_\theta(EX) = t(EX) = F_\theta(EX) = F(EX) \geq \kappa$ as $|D| = \kappa$. We have $2^{2^{F_\theta(X)}} = 2^{2^\omega} = 2^{\mathfrak{c}}$ but $F_\theta(EX) \geq \kappa$ for any cardinal κ.
When $\kappa = (2^{\mathfrak{c}})^+$ we have that

- $F_\theta(EX) \geq (2^{\mathfrak{c}})^+ > 2^{\mathfrak{c}} = 2^{2^{F_\theta(X)}}$.

- $t_\theta(EX) \geq (2^{\mathfrak{c}})^+ > 2^{\mathfrak{c}} = 2^{2^{t_\theta(X)}}$.

Proposition 17. Let X be a space

(a) $d(EX) \leq d(X_s) \leq d(X)$;

(b) If X is T_3, then $d(EX) = d(X)$;

(c) $|EX| \leq 2^{2^{d(X_s)}}$ and $|EX| \leq d(X_s)^{\chi(EX)}$;

(d) If X is separable, then EX is separable, $|EX| \leq 2^{\mathfrak{c}}$ and $t(EX) \leq \mathfrak{c}$;

(e) $w(EX) \leq 2^{w(X)}$ and $w(PX) \leq 2^{w(X)}$.

Proof.
(a) The inequality $d(X_s) \leq d(X)$ is clear. To prove the inequality $d(EX) \leq d(X_s)$, let D be a dense subset in X_s such that $d(X_s) = |D|$. For each $d \in D$, select $x_d \in k_X^{\leftarrow}(d)$ and let $D' = \{x_d : d \in D\}$. We have D' is dense in EX and $|D'| = |D| = d(X_s)$. Then, $d(EX) \leq |D'| = d(X_s)$.
(b) Suppose X is T_3 and D be a dense subset in EX such that $d(EX) = |D|$. The map $k_X : EX \to X$ is continuous and onto. Then, $k_X(D)$ is dense in X and $d(X) \leq |k_X(D)| \leq |D| = d(EX)$. Thus, with (a) we have that $d(EX) = d(X)$.
(c) Recall two well-known results by Pospišil ([18]): "If X is Hausdorff, then $|X| \leq 2^{2^{d(X)}}$ and $|X| \leq d(X)^{\chi(X)}$". So, $|EX| \leq 2^{2^{d(EX)}}$ and, by (a), $|EX| \leq 2^{2^{d(X_s)}}$. Also, $|EX| \leq d(EX)^{\chi(EX)}$ and, by (a), $|EX| \leq d(X_s)^{\chi(EX)}$.
(d) If X is separable, then $d(EX) \leq d(X_s) \leq d(X) \leq \omega$. Thus, by (c), $|EX| \leq 2^{2^{\omega}} = 2^{\mathfrak{c}}$ and moreover EX is separable too. Also, as $EX \subseteq \beta\omega$, then $t(EX) \leq t(\beta\omega) = \mathfrak{c}$.
(e) First note that $o(X) = |\tau(X)| \leq 2^{w(X)}$. A base for EX is $\{oU : U \in \tau(X)\}$; so, $w(EX) \leq o(X)$. Likewise, a base for PX is $\{oU \cap k_X^{\leftarrow}(V) : U, V \in \tau(X)\}$; so, $w(PX) \leq o(X)$. □

Lemma 18. Let X be a Hausdorff space and $(p_n)_{n\in\omega}$ a sequence converging to $p \in X$ where $p_n \neq p$ for $n \in \omega$. Then, $|k_X^{\leftarrow}(p)| \geq 2^{\mathfrak{c}}$.

Proof. As X is Hausdorff, there is an open set $U_1 \in \tau(X)$ such that $p_{n_1} \in U_1$ and $p \notin cl_X U_1$. So, let $n_2 = \inf\{m : p_m \in X \setminus cl_X U_1\}$. Then, there is an open set $U_2 \in \tau(X)$ such that $p_{n_2} \in U_2$, $p \notin cl_X U_2$ and $U_2 \subseteq X \setminus cl_X U_1$. So, let $n_3 = \inf\{m : p_m \in X \setminus (cl_X U_1 \cup cl_X U_2)\}$. Then, there is an open set $U_3 \in \tau(X)$ such that $p_{n_3} \in U_3$, $p \notin cl_X U_3$ and $U_3 \subseteq X \setminus (cl_X U_1 \cup cl_X U_2)$. Continuing by induction we obtain a subsequence $(q_n)_{n\in\omega}$ of $(p_n)_{n\in\omega}$ and a family of pairwise disjoint sets $\{U_n : n \in \omega\}$ such that $q_n \in U_n$ and $p \notin cl_X U_n$ for $n \in \omega$. Now, let $\mathcal{U} \in \beta\omega \setminus \omega$ and for $A \in \mathcal{U}$, let $U_A = \bigcup_{n\in A} U_n$ and $\mathcal{F}_{\mathcal{U}} = \{U_A : A \in \mathcal{U}\}$. Note that
(i) $\mathcal{F}_{\mathcal{U}}$ is an open filterbase. If $A, B \in \mathcal{U}$, $U_A \cap U_B = U_{A\cap B}$ and $A \cap B \in \mathcal{U}$.
(ii) $a_X(\mathcal{F}_{\mathcal{U}}) = \{p\}$. Let $T \in T(X)$ and $p \in T$; there exists $m \in \omega$ such that $\{q_n : n \geq m\} \subseteq T$. In particular $T \cap U_n \neq \emptyset$ for all $n \geq m$. If $A \in \mathcal{U}$, A is infinite subset of ω and $\{q_n : n \in A\} \cap T \neq \emptyset$. Therefore $T \cap U_A \neq \emptyset$.
(iii) Let $\mathcal{G}_{\mathcal{U}}$ be an open ultrafilter on X such that $\mathcal{G}_{\mathcal{U}} \supseteq \mathcal{F}_{\mathcal{U}} \cup \mathcal{N}_p^o$. So, $a_X(\mathcal{G}_{\mathcal{U}}) = c_X(\mathcal{G}_{\mathcal{U}}) = \{p\}$ ($\mathcal{G}_{\mathcal{U}} \in EX$).
(iv) Let $\mathcal{V} \in \beta\omega \setminus \omega$ and $\mathcal{V} \neq \mathcal{U}$. Since $\mathcal{V} \neq \mathcal{U}$, there are $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that $A \cap B = \emptyset$. Now, $U_A \cap U_B \neq \emptyset$ and $U_A \in \mathcal{F}_{\mathcal{U}} \subseteq \mathcal{G}_{\mathcal{U}}$ and $U_B \in \mathcal{F}_{\mathcal{V}} \subseteq \mathcal{G}_{\mathcal{V}}$. Thus, $\mathcal{F}_{\mathcal{U}} \neq \mathcal{G}_{\mathcal{V}}$.

(v) Now, consider the following map $\beta\omega \setminus \omega \to k_X^{\leftarrow}(p)$ defined by $\mathcal{U} \mapsto \mathcal{F}_{\mathcal{U}}$. This function in 1-to-1 and thus $|k_X^{\leftarrow}(p)| \geq |\beta\omega \setminus \omega| = 2^{\mathfrak{c}}$. $\square$

Proposition 19. Let X be a Hausdorff space containing a convergent sequence $(p_n)_{n\in\omega} \to p$ where $p_n \neq p \in X$ for $n \in \omega$. Then,

(a) $\beta\omega \hookrightarrow k_X^{\leftarrow}(p) \subseteq EX$ and $|EX| \geq 2^{\mathfrak{c}}$;

(b) For $\kappa \geq \omega$, $\beta\omega \hookrightarrow EX^{\kappa}$;

(c) $F(PX) \geq F(EX) \geq \mathfrak{c}$, $t(EX) \geq \mathfrak{c}$ and $t(PX) \geq \mathfrak{c}$.

Proof.
(a) By previous lemma, $k_X^{\leftarrow}(p)$ is infinite.
By *Proposition 13(f)*, $\beta\omega \hookrightarrow k_X^{\leftarrow}(p)$. Also, we have that $|EX| \geq 2^{\mathfrak{c}}$.
(b) Let $f \in X^{\kappa}$ be defined by $f(\alpha) = p$ for $\alpha < \kappa$.
Also, let $f_n(\alpha) = \begin{cases} p & \text{if } \alpha \neq 0, \\ p_n & \text{if } \alpha = 0. \end{cases}$ Then $f_n \to f$ in X^{κ}.
By (a), $\beta\omega \hookrightarrow k_{X^{\kappa}}^{\leftarrow}(f) \subseteq EX^{\kappa}$.
(c) As $k_X^{\leftarrow}(p)$ is infinite, by (a), $\beta\omega \hookrightarrow k_X^{\leftarrow}(p)$. So, $\mathfrak{c} = F(\beta\omega) \leq F(k_X^{\leftarrow}(p)) \leq F(EX)$. In similar way, we have that $\mathfrak{c} = t(\beta\omega) \leq t(k_X^{\leftarrow}(p)) \leq t(EX)$, $\mathfrak{c} = F(\beta\omega) \leq F(\Pi_X^{\leftarrow}(p)) \leq F(PX)$ and $\mathfrak{c} = t(\beta\omega) \leq t(\Pi_X^{\leftarrow}(p)) \leq t(PX)$. $\square$

Corollary 20. Let X be a second countable space. We have that:

(a) If X is discrete, $EX = PX = X$ and $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = F_\theta(PX) = F(PX) = t_\theta(PX) = t(PX) = \omega$;

(b) If X is not discrete, $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = F_\theta(PX) = F(PX) = t_\theta(PX) = t(PX) = \mathfrak{c}$.

Proof.
(a) As X is second countable and discrete, $X = \mathbb{N} = \omega$ that is ED and semiregular. By *Proposition 13(b)*, we have that $PX = EX \simeq X$.
(b) Follows from *Propositions 17(e)* and *19(c)*. $\square$

Now, in the same way as seen for H-closed (Urysohn) spaces in the previous section, we want to study the relationships among the cardinal functions $F_\theta(EX)$, $F_\theta(PX)$, $t_\theta(EX)$ and $t_\theta(PX)$.

Example 21. For the well-known spaces of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{J}$ and $\mathbb{R}$, we have the following results as a consequence of the previous corollary:

(a) $F_\theta(E\mathbb{N}) = F(E\mathbb{N}) = t_\theta(E\mathbb{N}) = t(E\mathbb{N}) =$
$= F_\theta(P\mathbb{N}) = F(P\mathbb{N}) = t_\theta(P\mathbb{N}) = t(P\mathbb{N}) = \omega$;

(b) $F_\theta(E\mathbb{Q}) = F(E\mathbb{Q}) = t_\theta(E\mathbb{Q}) = t(E\mathbb{Q}) =$
$= F_\theta(P\mathbb{Q}) = F(P\mathbb{Q}) = t_\theta(P\mathbb{Q}) = t(P\mathbb{Q}) = \mathfrak{c}$;

(c) $F_\theta(E\mathbb{J}) = F(E\mathbb{J}) = t_\theta(E\mathbb{J}) = t(E\mathbb{J}) =$
$= F_\theta(P\mathbb{J}) = F(P\mathbb{J}) = t_\theta(P\mathbb{J}) = t(P\mathbb{J}) = \mathfrak{c}$;

(d) $F_\theta(E\mathbb{R}) = F(E\mathbb{R}) = t_\theta(E\mathbb{R}) = t(E\mathbb{R}) =$
$= F_\theta(P\mathbb{R}) = F(P\mathbb{R}) = t_\theta(P\mathbb{R}) = t(P\mathbb{R}) = \mathfrak{c}$.

Now, our next goal is to compute the cardinal functions F_θ and t_θ for $E\mathbb{I}^\kappa$ ($\mathbb{I}^\kappa$ is the *Tychonoff cube of weight* κ) and $E\mathbb{D}^\kappa$ ($\mathbb{D}^\kappa$ is the *Cantor cube of weight* κ) where $\kappa \geq \omega$.

The function $f : \mathbb{D}^\omega \to \mathbb{I}$ defined by $x \mapsto \sum_{i\in\omega} \frac{x(i)}{2^{i+2}}$ is a continuous closed and open surjection (see *4.3* in [10]), By *6.5(c)* in [22], there is a closed subset $A \subseteq \mathbb{D}^\omega$ such that $f|_A : A \to \mathbb{I}$ is an irreducible perfect surjection.
Now, by *Lemma 9*, A has no isolated points. So, A is second countable and compact Hausdorff. By *3.3(e)* in [22], A is homeomorphic to $\mathbb{D}^\omega$. Thus, we have a continuous, perfect irreducible surjection $g : \mathbb{D}^\omega \to \mathbb{I}$.

Lemma 22. ([22]) Let X and Y be spaces and $f : X \to Y$ be perfect, irreducible, θ-continuous surjection. Then, EX and EY are homeomorphic.

Theorem 23. *For $\kappa \geq \omega$, $E\mathbb{D}^\kappa$ and $E\mathbb{I}^\kappa$ are homeomorphic.*

Proof. By *Lemma 8*, there is a continuous, perfect irreducible surjection $\varphi : \mathbb{D}^\kappa \to \mathbb{I}^\kappa$. Then, by previous lemma, $E\mathbb{D}^\kappa$ and $E\mathbb{I}^\kappa$ are homeomorphic. □

Proposition 24. For $\kappa \geq \omega$, we have that:

(a) $\beta\omega \hookrightarrow E\mathbb{D}^\kappa$;

(b) $d(E\mathbb{D}^\kappa) = d(\mathbb{D}^\kappa) = \log\kappa = \inf\{\lambda : 2^\lambda \geq \kappa\}$;

(c) $\kappa \leq F(\mathbb{D}^\kappa) \leq F(E\mathbb{D}^\kappa) = t(E\mathbb{D}^\kappa) \leq 2^{d(E\mathbb{D}^\kappa)} = 2^{\log\kappa}$.

Proof.
(a) By *Proposition 19(b)*.
(b) As $\mathbb{D}^\kappa$ is T_3, by *Proposition 17(b)*, we have that $d(E\mathbb{D}^\kappa) = d(\mathbb{D}^\kappa) =$ (by *11.8(d)* in [16]) $= \inf\{\lambda : 2^\lambda \geq \kappa\}$.
(c) $\kappa =$ (by *Proposition 13(a)* in [7]) $= t(\mathbb{D}^\kappa) =$ (by *Theorem 4(a)*) $= F(\mathbb{D}^\kappa) \leq$ (as F is hereditary closed) $\leq F(E\mathbb{D}^\kappa) =$ (by *Theorem 4(a)*) $= t(E\mathbb{D}^\kappa) \leq$ (by *Proposition 14(b)*) $\leq 2^{d(E\mathbb{D}^\kappa)} \leq$ (by *(a)*) $\leq 2^{\log\kappa}$. □

Corollary 25. Suppose [**GCH**].
If κ is a successor cardinal, then $t(E\mathbb{D}^\kappa) = F(E\mathbb{D}^\kappa) = \kappa$.

Proof. If $\kappa = \mu^+$, $\log \kappa \leq \mu$ and $2^{\log \kappa} \leq 2^\mu$. By [**GCH**], $2^\mu = \mu^+ = \kappa$.
So, $\kappa \leq t(E\mathbb{D}^\kappa) \leq 2^{\log \kappa} \leq 2^\mu = \kappa$. □

Proposition 26. If $\omega \leq \kappa \leq \mathfrak{c}$, then $t(E\mathbb{D}^\kappa) = F(E\mathbb{D}^\kappa) = \mathfrak{c}$.

Proof. Let $\omega \leq \kappa \leq \mathfrak{c}$. By the *Hewitt-Marczewski-Pondiczery Theorem*, $\mathbb{D}^\kappa$ is separable as $\kappa \leq \mathfrak{c}$. By *Proposition 17(d)*, $E\mathbb{D}^\kappa$ is separable too and $t(E\mathbb{D}^\kappa) \leq \mathfrak{c}$. Now, by *Proposition 24(a)*, $\beta\omega \hookrightarrow E\mathbb{I}^\kappa$. So, $\mathfrak{c} = t(\beta\omega) \leq t(E\mathbb{I}^\kappa) =$ (by *Theorem 23*) $= t(E\mathbb{D}^\kappa)$. □

Corollary 27.
$t(E\mathbb{D}^\omega) = t(E\mathbb{D}^{\omega_1}) = t(E\mathbb{D}^\mathfrak{c}) = F(E\mathbb{D}^\omega) = F(E\mathbb{D}^{\omega_1}) = F(E\mathbb{D}^\mathfrak{c}) = \mathfrak{c}$.

Note 28.
(*a*) [$\neg$**CH**] $t(E\mathbb{D}^{\omega_2}) = F(E\mathbb{D}^{\omega_2}) = \mathfrak{c} \geq \omega_2$.
(*b*) [**GCH**] $t(E\mathbb{D}^{\omega_2}) = F(E\mathbb{D}^{\omega_2}) = \omega_2 > \mathfrak{c} = \omega_1$.

The next result is known as the ω-version of the combinatorial principle due to Tarski (for details see *2.1* in [16]).

Lemma 29. There is a family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ such that $|\mathcal{A}| = \mathfrak{c}$, $|A_\alpha| = \omega$ for each $A_\alpha \in \mathcal{A}$ (with $\alpha \in \mathfrak{c}$), and the intersection of any two distinct elements of $\mathcal{A}$ is finite (i.e. $|A_\alpha \cap A_\beta| < \omega$ for $\alpha < \beta < \mathfrak{c}$).

We have the following questions:

Question 30.
Is it true that for a space X, $F_\theta(EX) \leq t_\theta(EX)$?

Example 31. By *Proposition 13(b)*, the previous question can be reformulate as follow: "Let E be an ED and semiregular space. Is $F(E) \leq t(E)$?". The answer is NO because if we consider $\beta\omega$, by previous lemma (with ω_1 instead of $\mathfrak{c}$), we can find $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ such that $A_\alpha \cap A_\beta$ is finite whenever $\alpha < \beta < \omega_1$. Now, we have that $cl_{\beta\omega}A_\alpha \cap cl_{\beta\omega}A_\beta \subseteq \omega$ is finite. So, for each $\alpha < \omega_1$ we select a point $p_\alpha \in cl_{\beta\omega}A_\alpha \setminus \omega$ and let $E = \omega \cup \{p_\alpha : \alpha < \omega_1\}$ (it is easy to see that $|E| = \omega_1$ and E is ED, semiregular). Also, $\omega \subseteq E \subseteq \beta\omega$ and $E \setminus \omega = \{p_\alpha : \alpha < \omega_1\}$ is closed. Moreover, as $cl_{\beta\omega}A_\alpha$ is clopen in $\beta\omega$, $cl_{\beta\omega}A_\alpha \cap E = \{p_\alpha\} \cup A_\alpha$. Then, $E \setminus \omega$ is closed and discrete and therefore $F(E) = |E \setminus \omega| = \omega_1$. To compute $t(E)$, let $B \subseteq E$

and $p \in cl_E B \setminus B$ and, as every point of ω is isolated, $p \notin \omega$ and therefore $p \in E \setminus \omega$. So, $p = p_\alpha$ for some $\alpha < \omega_1$ and we note $p_\alpha \in cl_{\beta\omega} A_\alpha$. So, $cl_{\beta\omega} A_\alpha \cap B \subseteq \omega$ and finally $t(E) = \omega$.

Question 32.
Is it true that for a space X, $t_\theta(EX) \leq F_\theta(EX)$?

Example 33. By *Proposition 13(b)*, the previous question can be reformulate as follow: "Let E' be an ED and semiregular space. Is $t(E') \leq F(E')$?". We do not know the answer but, as seen in *Examples 11* and *15* in [7], we have this partial answer in which the space is ED but not semiregular.
Starting with $\mathbb{I} = [0,1]$ where $\tau(\mathbb{I})$ is the usual topology, we consider the space $Y = E[0,1]$ with this finer topology:

$$\tau(E[0,1]) \text{ is generated by } \mathcal{B} = \{oU \setminus F : U \in \tau(\mathbb{I}), F \in [E\mathbb{I}]^{\leq \mathfrak{c}}\}.$$

We start with this result:

Claim 1. $cl_Y(oU \setminus F) = oU$.

Proof. As oU (basic open set in $E\mathbb{I}$) is closed in $E\mathbb{I}$ (with $|oU| = 2^{\mathfrak{c}}$), oU is closed in Y. Let $\mathcal{U} \in oU$, then $U \in \mathcal{U}$. Now, let $\mathcal{U} \in oV \setminus G$ where $G \in [E\mathbb{I}]^{\leq \mathfrak{c}}$. Then, $V \in \mathcal{U}$, $U \cap V \in \mathcal{U}$ and $(oU \setminus F) \cap (oV \setminus G) = (oU \cap oV) \setminus (F \cup G) = o(U \cap V) \setminus (F \cup G)$. Now, $F \cup G \in [E\mathbb{I}]^{\leq \mathfrak{c}}$ and then $o(U \cap V) \setminus (F \cup G) \neq \emptyset$. Thus, $p \in cl_Y(oU \setminus F)$ and $cl_Y(oU \setminus F) = oU$. $\square$

Now, we have $Y_s = E\mathbb{I}$ and note that Y is ED, H-closed and Urysohn (but not semiregular). Also, by *Theorem 4(b)*, $F_\theta(Y) = F_\theta(Y_s) = F(Y_s) = F(E\mathbb{I}) = \mathfrak{c}$ and $t_\theta(Y) = t_\theta(Y_s) = t(Y_s) = t(E\mathbb{I}) = \mathfrak{c}$. So, it remains only to calculate $F(Y)$ and $t(Y)$. At first, we have that $\mathfrak{c} = F_\theta(Y) \leq F(Y)$. Consider $\beta\omega \subseteq E\mathbb{I} \subseteq \beta\omega$ and $hL(\beta\omega) \leq w(\beta\omega) = \mathfrak{c}$. So, $hL(E\mathbb{I}) = \mathfrak{c}$ and for each $B \subseteq E\mathbb{I}$, $L(B) \leq \mathfrak{c}$. Now let $\{x_\alpha\}_{\alpha \in \mathfrak{c}^+}$ be a free sequence in Y. Call $B = \{x_\alpha\}_{\alpha \in \mathfrak{c}^+}$, then $L(B) \leq \mathfrak{c}$. On the other hand, we start with this result:

Claim 2. There is a point $b \in B$ such that if $b \in oU \in \tau(E\mathbb{I})$ then $|oU \cap B| = \mathfrak{c}^+$ (b is a complete accumulation point).

Proof. Assume the contrary: for each $p \in B$, there exists $oU_p \in \tau(E\mathbb{I})$ such that $p \in oU_p$ and $|oU_p \cap B| \leq \mathfrak{c}$. The open cover $\{oU_p\}_{p \in B}$ of B has a subcover of size $\mathfrak{c}$ and there is a subset $A \subset B$ with $|A| = \mathfrak{c}$ such that $B \subseteq \bigcup_{p \in A} oU_p$. Then $B = \bigcup_{p \in A}(oU_p \cap B)$ and, as A and $oU_p \cap B$ have size $\mathfrak{c}$, B have cardinality $\mathfrak{c}$. But this is not possible as $|B| = \mathfrak{c}^+$. $\square$

By Claim 2, there is a $\beta < \mathfrak{c}^+$ such that x_β is a complete accumulation of B in $E\mathbb{I}$. Consider $B' = \{x_\alpha : \alpha > \beta\}$ and then $x_\beta \notin cl_Y B'$. There exists

$U \in \tau(\mathbb{I})$ and $F \in [E\mathbb{I}]^{\leq \mathfrak{c}}$ such that $(oU \setminus F) \cap B' = \emptyset$ and $x_\beta \in oU \setminus F$. Also, $(oU \setminus F) \cap B \subseteq B \setminus B' = \{x_\alpha : \alpha \leq \beta\}$. AS $|oU \cap B| = \mathfrak{c}^+$, $oU \cap B \subseteq ((oU \setminus F) \cap B) \cup F = \{x_\alpha : \alpha \leq \beta\} \cup F$ where $|\{x_\alpha : \alpha \leq \beta\}| = \mathfrak{c}$ and $|F| = \mathfrak{c}$. This contradicts B is a free sequence in Y and completes the proof that $F(Y) = \mathfrak{c}$.
Now, let $B \subseteq Y$ where $B = oU \setminus \{p\}$ with $p \in oU$. It easy to see that $p \in cl_Y B$ and let $G \in [B]^{\leq \mathfrak{c}}$, $(oU \setminus G) \cap G \neq \emptyset$ and $p \notin cl_Y G$. Then, $t(Y) \geq \mathfrak{c}^+$. On the other hand, let $A \subseteq Y$ with $p \in cl_Y A \setminus A$. Suppose $|A| > \mathfrak{c}^+$ and, for each $U \in \tau(\mathbb{I})$ and $p \in oU$, $|oU \cap A| \geq \mathfrak{c}^+$. Now, let $B_U \subseteq oU \cap A$ such that $|B_U| = \mathfrak{c}^+$ and consider $B = \bigcup_{U \in \tau(\mathbb{I}), p \in oU} B_U$. We have that $B \subseteq A$, $|B| = \mathfrak{c}^+$ and $p \in cl_Y B$. Thus, $t(Y) = \mathfrak{c}^+$.

Question 34.
– *Is it true that for a space X, $F_\theta(PX) \leq t_\theta(PX)$?*
– *Is it true that for a space X, $t_\theta(PX) \leq F_\theta(PX)$?*

Example 35. The previous questions can be reformulate as follow: "Let E be an ED space. Is $F(E) \leq t(E)$? Is $t(E) \leq F(E)$?". The answer is NO in both cases: for the first question see *Example 31* and for the second question see *Example 33*.

Moreover, it is interesting to see the following example for ED spaces:

Example 36.
For $X = \kappa\omega$, X is ED, $PX = X$, $EX = X_s = \beta\omega$, and $\mathfrak{c} = F_\theta(X) < F(X) = 2^{\mathfrak{c}}$.
For $Y = (E[0,1], \tau(\mathcal{B}))$ constructed in *Example 33*, Y is ED, $PY = Y$, $EY = Y_s = E\mathbb{I}$, and $\mathfrak{c} = t_\theta(Y) < t(Y) = \mathfrak{c}^+$.

Question 37. *By Corollary 20, we know that for second countable space X, $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = F_\theta(PX) = F(PX) = t_\theta(PX) = t(PX) = \kappa$ where κ is ω (if X is discrete) or $\mathfrak{c}$ (if X is not discrete).*
So, the natural question is whether there is a non second countable space X for which $F_\theta(EX) = F(EX) = t_\theta(EX) = t(EX) = F_\theta(PX) = F(PX) = t_\theta(PX) = t(PX) = \omega_1$.

Example 38. Similarly to what we have seen in *Example 31*, if we consider the space $E = \omega \cup \{p_\alpha : \alpha < \omega_1\}$ we may note that $\omega \subseteq E \subseteq \beta\omega$ and let $B = E \setminus \omega$ we note that it has a complete accumulation point $q \in \beta\omega$. Also, if $q \in U \in CLOP(\beta\omega)$ then, $|U \cap B| = \omega_1$. Then $U = cl_{\beta\omega}(U \cap \omega)$

and in particular $U \cap E = cl_{\beta\omega}(U \cap \omega) \cap E = cl_{\beta\omega}(U \cap \omega)$. Now, let $E' = E \cup \{q\} \subseteq \beta\omega$ (it is easy to see that $|E'| = \omega_1$ and E' is ED, semiregular). Also, $F(E') = t(E') = \omega_1$.

We conclude this paper with this interesting result:

Example 39. In this example our goal is to determine the behavior of the cardinal functions F, F_θ, t and t_θ for the well-known *Urysohn's H-closed Example* (see [25] or *4.8(d)* in [22] for details) and for its Iliadis and Banaschewski absolutes.
Let $Y = \mathbb{N} \times \mathbb{Z} \subseteq \mathbb{R}^2$ with the subspace topology inherited from the usual topology on the plane $\mathbb{R}^2$. Now, let $X = Y \cup \{p, q\}$ where $p = (0, 1)$ and $q = (0, -1)$.
A subset $U \subseteq X$ is defined to be open if:

(i) $p \in U$, then there is $m \in \mathbb{N}$ such that $\{n\} \times \mathbb{N} \subseteq U$ for all $n \geq m$ or

(ii) $q \in U$, then there is $m \in \mathbb{N}$ such that $\{n\} \times \mathbb{N}^- \subseteq U$ for all $n \geq m$ or

(iii) $(n, 0) \in U$, then there is $m \in \mathbb{N}$ such that $(n, k) \in U$ for all $|k| \geq m$.

Thus, all the points of $D = X \setminus ((\mathbb{N} \times \{0\}) \cup \{p, q\})$ are isolated.

The space X is *H-closed*, *semiregular*, *minimal Hausdorff* but not compact nor Urysohn. Also, as all open sets in X are countable, we have that $F(X) = F_\theta(X) = t(X) = t_\theta(X) = \omega$.

Also, notice that the space X is second countable but not discrete. Thus, by *Corollary 20(b)* we have that $F(EX) = F_\theta(EX) = F(PX) = F_\theta(PX) = t(EX) = t_\theta(EX) = t(PX) = t_\theta(PX) = \mathfrak{c}$.

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References

[1] A.V. Arhangel'skiĭ, *On bicompacta hereditarily satisfying Suslin's condition. Tightness and free sequences*, **Soviet Math. Dokl.**, **12** (1971), 1253–1257.

[2] A.V. Arhangel'skiĭ, *Structure and classification of topological spaces and cardinal invariants*, **Russian Math. Surveys**, **33** (1978), 33–96.

[3] A.V. Arhangel'skiĭ and V.I. Ponomarev, *Fundamentals of General Topology: Problems and Exercises*, **D. Reidel Publ. Comp. - Kluwer Acad. Publ.**, Dordrecht, 1984.

[4] B. Balcar and F. Franěk, *Independent families in complete Boolean algebras*, **Trans. Amer. Math. Soc.**, **274** (1982), 607–618.

[5] B. Banaschewski, *Projective covers in categories of topological spaces and topological algebras*, Proc. "Kanpur Topological Conference", Prague 1968, **Academic Press**, (1971), 63–91.

[6] I. Bandlov and V.I. Ponomarev, *Cardinal invariants of tightness type, absolutes of spaces and continuous maps*, **Russian Math. Surveys**, **35**:3 (1980), 25–33.

[7] F. Cammaroto, A. Catalioto and J. Porter, *Cardinal functions* $F_\theta(X)$ *and* $t_\theta(X)$ *for H-closed spaces*, submitted.

[8] F. Cammaroto and Lj.D. Kočinac, *On* θ*-tightness*, **Facta Universitatis (Niš), Ser. Math. Inform. 8** (1993), 77–85.

[9] F. Cammaroto and Lj.D. Kočinac, *A note on* θ*-tightness*, **Rend. Circ. Mat. Palermo, Ser II** 42 (1993), 129–134.

[10] J. Dugundji, *Topology*, **Allyn and Bacon**, Boston, 1966.

[11] *Encyclopedia of General Topology* (K.P. Hart, J. Nagata and J.E. Vaughan eds.), **Elsevier**, 2003.

[12] R. Engelking, *General Topology*, **Heldermann-Verlag**, Berlin, 1989.

[13] J. Flachsmeyer, *Topologische Projektivräume*, **Math. Nachr.**, **26** (1963), 57–66.

[14] L. Gillman and M. Jerison, *Rings of continuous functions*, **Springer-Verlag**, 1976.

[15] A. Gleason, *Projective Topological Spaces*, **Ill. J. Math.**, **2** (1958), 482–489.

[16] R. Hodel, *Cardinal Functions I*, In: 'Handbook of Set-theoretic Topology' (K. Kunen, J.E. Vaughan, eds.), pp. 1–61, **North-Holland**, Amsterdam, 1984.

[17] S. Iliadis, *Absolutes of Hausdorff spaces*, **Soviet Math. Dokl.**, **4** (1963), 295–298.

[18] I. Juhász, *Cardinal Functions in Topology - Ten Years Later*, **Math. Centre Tracts** 123, Amsterdam, 1980.

[19] J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, **Dissert. Math.**, **66** (1969), 1–55.

[20] V.I. Ponomarev, *The absolute of a topological space*, **Soviet Math. Dokl.**, **4** (1963), 299–302.

[21] V.I. Ponomarev and L.B. Shapiro, *Absolutes of topological spaces and their continuous maps*, **Russian Math. Surveys**, **31**:5 (1976), 138–154.

[22] J.R. Porter and R.G. Woods, *Extensions and absolutes of Hausdorff spaces*, **Springer-Verlag**, New York, 1988.

[23] M.H. Stone, *Applications of Boolean Algebras to Topology*, **Mat. Sb.**, **1** (1936), 765-771.

[24] M.H. Stone, *Applications of the Theory of Boolean Rings to General Topology*, **Trans. Amer. Math. Soc.**, **41** (1937), 375–481.

[25] P. Urysohn, *Über die Mächtigkeit der zusammenhängenden Mengen*, **Math. Ann.**, **94** (1925), 262–295.

[26] R.G. Woods, *A survey of absolutes of topological spaces*, Topological *Structures II*, **Math. Centre Tracts**, **116** (1979), 323–362.

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